On the Duality Operator of a Convex Cone

Bit-Shun Tam*
Department of Mathematics
Tamkang University
Tamsui, Taipei, Taiwan 251, Republic of China

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ABSTRACT

Let C be a convex set in R^n. For each $y \in C$, the cone of C at y, denoted by cone(y, C), is the cone $\{\alpha(x-y): \alpha \geqslant 0 \text{ and } x \in C\}$. If K is a cone in R^n, we shall denote by K^* its dual cone and by $\mathscr{F}(K)$ the lattice of faces of K. Then the duality operator of K is the mapping $d_K : \mathscr{F}(K) \to \mathscr{F}(K^*)$ given by $d_K(F) = (\operatorname{span} F)^{\perp} \cap K^*$. Properties of the duality operator d_K of a closed, pointed, full cone K have been studied before. In this paper, we study d_K for a general cone K, especially in relation to $d_{\text{cone}(y,K)}$, where $y \in K$. Our main result says that, for any closed cone K in R^n, the duality operator d_K is injective (surjective) if and only if the duality operator $d_{\text{cone}(y,K)}$ is injective (surjective) for each vector $y \in K \sim [K \cap (-K)]$. In the last part of the paper, we obtain some partial results on the problem of constructing a compact convex set C, which contains the zero vector, such that cone$(0, C)$ is equal to a given cone.

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1. INTRODUCTION

Let K be a convex cone in the euclidean space R^n. Denote by $\mathscr{F}(K)$ the lattice of faces of K, and by K^* its dual cone in R^n. In recent years, a lot of work has been done on the face lattice $\mathscr{F}(K)$ when K is a closed, pointed cone (see Barker [1, 3, 4, 5] and Loewy and Tam [9]). The relations between the lattices $\mathscr{F}(K)$ and $\mathscr{F}(K^*)$ for a proper cone K have also been investigated independently by Barker [2, 3] and Tam [14] through a study of the

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duality operators between them. Nevertheless, there are still many unsettled simple natural questions about the duality operator of a proper cone. (See the open problems 2, 3, and 4 in our last section.) The present work suggests that, to tackle these problems, it may be helpful to consider the points' cones of a cone. Our interest in the duality operators rests on the fact that in the study of the set $\pi(K)=\{A\in R^{n\times n}: AK\subseteq K\}$ as a cone, or as a semigroup, or as a semiring, the duality operators have proved to be useful in formulating many results (see Tam [13, 14]).

In Section 2 we give the basic properties of the duality operator of a general cone. It will be shown that the duality operator of a general cone is not so well behaved as that of a proper cone. Even though our primary interest is on proper cones, general cones are also considered here, because when we come to consider the points' cones, nonclosed cones will be involved.

If y is a point of a convex set C, then the cone of C at y, denoted by cone(y,C), is the cone $\{\alpha(x-y): \alpha\geqslant 0 \text{ and } x\in C\}$. Section 3 is devoted to a study of the points' cones cone(y,K) of a cone K. In particular, we prove that the face lattice of cone(y,K) is isomorphic to the interval sublattice $[\varphi(y),K]$ of $\mathscr{F}(K)$.

In Section 4 we show that the duality operator of a cone K and those of its points' cones cone(y,K) usually share common special properties. For instance, if K is closed and pointed, then the duality operator of K is injective (surjective) if and only if the duality operator of cone(y,K) is injective (surjective) for each nonzero vector y of K. Thus the problem of determining the injectivity or the surjectivity of the duality operator of a cone can be reduced to the corresponding problems for its nontrivial points' cones, which are essentially of smaller dimension.

In Section 5 we are concerned with the problem of constructing a compact convex set in R^n, which contains the origin, such that its point's cone at the origin is equal to a prescribed cone. It is shown that such a compact convex set always exists when the prescribed cone is of some special types. However, the existence of such a compact convex set in the general case is still unknown. We believe that the study of this problem is worthwhile, as it suggests some ways to construct rich examples of finite dimensional compact convex sets (and hence closed, pointed cones).

In Section 6 we end with open problems and related examples.

2. PRELIMINARIES

A familiarity with elementary results on cones and convex sets is assumed. For references, the reader may consult [1], [5], [6], and [10]. To fix notation and terminology, we begin with some definitions.

A nonempty subset K of R^n is called a (*convex*) *cone* if for any $\alpha, \beta \geqslant 0$ and $x, y \in K$, we have, $\alpha x + \beta y \in K$. K is *closed* if it is closed in the usual topology of R^n. K is *pointed* if $K \cap (-K) = \{0\}$. If $\operatorname{int} K \neq \varnothing$, or equivalently, $K - K = R^n$, then K is called *full*. A closed, pointed full cone will be called a *proper* cone.

A subcone F of K is a *face* of K if $x, y \in K$ and $x + y \in F$ imply $x, y \in F$. Then we write $F \trianglelefteq K$. An intersection of faces of K is clearly also a face of K. If $S \subseteq K$ then the intersection of all faces of K containing S is called the *face of K generated by* S and is denoted by $\varphi(S)$. If $S = \{x\}$, we write $\varphi(x)$ for simplicity. Denote by $\mathscr{F}(K)$ the set of all faces of K. Then $\mathscr{F}(K)$ forms a complete lattice of finite length under the partial ordering set inclusion, with meet and join operations given by $F \wedge G = F \cap G$ and $F \vee G = \varphi(F \cup G)$. The concept of a face is usually defined for closed, pointed cones, but we use the same definition for general cones. Most of the known results about the faces of a closed, pointed cone (see Barker [1]) are still valid for a general cone. For example, if $x \in K$ then $\varphi(x) = \{y \in K : x - \alpha y \in K \text{ for some } \alpha > 0\}$; if $x \in K$ and $F \trianglelefteq K$, then $x \in \operatorname{relint} F$ iff $F = \varphi(x)$, where $\operatorname{relint} F$ denotes the interior of F relative to its own linear span. If $F, G \trianglelefteq K$ then $F \vee G = \varphi(F + G)$. It is easy to show that the smallest and the greatest element of the lattice $\mathscr{F}(K)$ are respectively L and K, where $L = K \cap (-K)$ is the *lineality space of* K. When K is closed and pointed, it is known that the face lattice $\mathscr{F}(K)$ is section complemented (see Loewy and Tam [9]); that is, for any $F_1 \trianglelefteq G \trianglelefteq K$, there exists $F_2 \trianglelefteq K$ such that $F_1 \wedge F_2 = \{0\}$ and $F_1 \vee F_2 = G$. The set $\{(\xi_1, \xi_2) \in R^2 : \xi_1 \geqslant 0 \text{ and } \xi_2 > 0\} \cup \{(0,0)\}$ serves as an example of a nonclosed cone whose face lattice is not section complemented, in fact not even complemented. But there are also nonclosed cones whose face lattices are relatively complemented. An example is provided by the cone$\{(\xi_2, \xi_2, \xi_3) \in R^3 : (\xi_1 + \xi_2^2)^{1/2} \leqslant \xi_3\} \sim \{\alpha(1,0,1) : \alpha > 0\}$, where $\sim$ denotes relative complement.

A cone K is called a *direct sum* of K_1 and K_2 and we write $K = K_1 \oplus K_2$ if (a) $K = K_1 + K_2$ and (b) $\operatorname{span} K_1 \cap \operatorname{span} K_2 = \{0\}$. Then K_1 and K_2 are themselves cones, and are also faces of K when K is pointed. Every cone K can be expressed as the direct sum of a linear subspace and a pointed cone, viz. $K = L \oplus (K \cap L^{\perp})$, where L is the lineality space of K and $L^{\perp}$ its orthogonal complement. Then the face lattices $\mathscr{F}(K \cap L^{\perp})$ and $\mathscr{F}(K)$ are isomorphic under the isomorphism $F \mapsto L \oplus F$. Thus the face lattice of a closed cone is always section complemented.

For any nonempty subset S of R^n, the set $S^* = \{z \in R^n : (z, y) \geqslant 0 \text{ for all } y \in S\}$ is called the *dual* of S (in R^n), where (z, y) denotes the usual inner product between the vectors z and y; S is always a closed cone, and is often called the *dual cone* of S when S is itself a cone. For properties of the dual of a set, the reader may consult Berman [6]. The concept of duality can be defined in a more general setting in terms of sets in a vector space and its dual space (see, for instance, Barker [2]). But, since we are working in finite dimensional real vector spaces, there is no loss of generality in restricting ourselves to euclidean spaces.

By the *duality operator* of a cone, we mean the mapping $d_K : \mathscr{F}(K) \to \mathscr{F}(K^*)$ given by $d_K(F) = (\operatorname{span} F)^\perp \cap K^*$. We shall call $d_K(F)$ the *dual face* of F. The concept of a dual face may be considered as a refinement of the concept of a dual cone. Similarly, we have a mapping $\delta_K^* : \mathscr{F}(K^*) \to \mathscr{F}(K)$ given by $\delta_K^*(G) = (\operatorname{span} G)^\perp \cap K$, which is usually different from d_{K^*}, the duality operator of K^*, unless K is closed. (Recall that $K^{**} = \operatorname{cl} K$.) A face F of K is said to be *exposed* if there exists $G \trianglelefteq K^*$ such that $\delta_K^*(G) = F$. K itself is always an exposed face of K, as $\delta_K^*(M) = K$, where M is the lineality space of K^*. Further, for any $F \trianglelefteq K$, $F \neq K$, we have that F is an exposed face of K if and only if F is the intersection of K with a (supporting) hypersubspace. For a proper cone K, the following result is known (see Tam [13, p. 7]): for any $F \trianglelefteq K$, $d_K(F) = \{0\}$ iff $F = K$; $d_K(F) = K^*$ iff $F = \{0\}$. When K is a general cone, the situation is more complicated.

PROPOSITION 2.1. *Let K be a cone in R^n. Denote by L the lineality space of K, and by M the lineality space of K^*. Then:*

(a) *$d_K(K) = M$ and $d_K(L) = K^*$.*
(b) *For any $F \trianglelefteq K$, if $d_K(F) = M$ then $F = K$.*
(c) *For any $G \trianglelefteq K^*$, $\delta_K^*(G) = K$ iff $G = M$.*
(d) *The following are equivalent statements:*
 (i) *L is an exposed face of K,*
 (ii) *For any $F \trianglelefteq K$, $d_K(F) = K^*$ implies $F = L$,*
 (iii) *$\delta_K^*(K^*) = L$,*
 (iv) *There does not exist a vector $x \in K$ such that $-x \in \operatorname{cl} K \sim K$.*

When K is closed, the above equivalent conditions are satisfied.

(e) *If K is closed, then for any $G \trianglelefteq K^*$, $\delta_K^*(G) = L$ implies $G = K^*$.*

Proof. For simplicity, we shall employ the notation $(z, S) \geqslant 0$ to mean "$(z, y) \geqslant 0$ for all $y \in S$." The notation $(z, S) = 0$, $(T, S) \geqslant 0$, etc. will have similar meanings.

(a): If $z \in d_K(K)$ then $(z, K) = 0 = (z, -K)$. Hence, $z \in K^* \cap (-K^*) = M$ and so $d_K(K) \trianglelefteq M$. But M is the smallest face of K^*; thus $d_K(K) = M$. Clearly, $(x, K^*) = 0$ for any $x \in L$. Thus $d_K(L) = K^*$.

(b): Suppose $F \trianglelefteq K$, $F \neq K$. Note that K is contained in $M^\perp$, as $(K, M) = 0$. By the standard separation theorem, there exists a hyperplane of $M^\perp$ which contains F such that K lies on one side of this hyperplane. As K

is a cone, it is easily shown that this hyperplane is in fact a hypersubspace of M. Hence, we can find a nonzero vector $z \in M^{\perp}$ such that $(z, K) \geqslant 0$ and $(z, F) = 0$. In other words, there exists a nonzero vector $z \in M^{\perp} \cap d_K(F)$. So $d_K(F) \neq M$.

(c): It is obvious that $(K, M) = 0$, so $\delta_K^*(M) = K$. Conversely, let $G \trianglelefteq K^*$ such that $\delta_K^*(G) = K$. Then $(G, K) = 0$, hence $G \subseteq K^* \cap (-K^*) = M$. Therefore, $G = M$.

(d), (i) $\Rightarrow$ (ii): Let $F \trianglelefteq K$ such that $d_K(F) = K^*$. Then $d_K(L) = K^* = d_K(F)$. As L is an exposed face of K, $L = \delta_K^* \circ d_K(L) = \delta_K^* \circ d_K(F) \trianglerighteq F$ [see Proposition 2.4(g), (d)]. Thus $F = L$.

(ii) $\Rightarrow$ (i): Suppose that L is not an exposed face of K. Then $\delta_K^* \circ d_K(L) \neq L$. But by Proposition 2.4(e), $d_K[\delta_K^* \circ d_K(L)] = d_K(L) = K^*$. So condition (ii) is not satisfied.

(iii) $\Leftrightarrow$ (i): $\delta_K^*(K^*) = L$ iff $\delta_K^* \circ d_K(L) = L$ iff L is an exposed face of K.

(i) $\Leftrightarrow$ (iv): First note that

$$\begin{aligned}
\delta_K^* \circ d_K(L) &= \delta_K^*(K^*) = K \cap (\operatorname{span} K^*)^{\perp} = K \cap (\operatorname{span} K^*)^* \\
&= K \cap [K^* + (-K^*)]^* = K \cap [K^{**} \cap (-K^{**})] \\
&= K \cap [\operatorname{cl} K \cap (-\operatorname{cl} K)] = K \cap (-\operatorname{cl} K).
\end{aligned}$$

Hence, L is an exposed face of K

iff $L = \delta_K^* \circ d_K(L)$
iff $K \cap (-K) = K \cap (-\operatorname{cl} K)$
iff there does not exist a vector $x \in K$ such that $-x \in \operatorname{cl} K \sim K$.

When K is closed, condition (iv) is clearly satisfied, and hence so are other equivalent conditions.

(e): If K is closed then $\delta_K^* = d_{K^*}$. Substituting K^* for K in (i), we obtain $\delta_K^*(M) = d_{K^*}(M) = (K^*)^* = K$. ■

Corollary 2.2. *Each maximal face of K is an exposed face of K.*

Proof. Suppose H is a maximal face of K which is nonexposed. Then $\delta_K^* \circ d_K(H)$ is a face of K properly containing H. So by the maximality of H, we have $\delta_K^*(d_K(H)) = K$. In view of Proposition 2.1 (c), we have $d_K(H) = M$, and by Proposition 2.1(b) we obtain $H = K$, which is a contradiction. ■

REMARK 2.3. By using the argument for computing $\delta_K^*(K^*)$ as in the proof of Proposition 2.1(d), (i) $\Leftrightarrow$ (iv), we can show that, for any $F \trianglelefteq K$ and $G \trianglelefteq K^*$, we have $d_K(F) = K^* \cap (-F^*)$ and $\delta_K^*(G) = K \cap (-G^*)$.

We collect below some of the basic properties of the mappings d_K and δ_K^*. In fact, we can state slightly more general results (cf. Barker [2, §§3, 4], Tam [13, Chapter 1, §§2, 3]).

PROPOSITION 2.4. *Let K_1 and K_2 be cones in R^n such that $(K_1, K_2) \geq 0$ (equivalently, $K_2 \subseteq K_1^*$, or $K_1 \subseteq K_2^*$). Let $\delta_1 : \mathscr{F}(K_1) \to \mathscr{F}(K_2)$ be the mapping defined by $\delta_1(F) = (\operatorname{span} F)^\perp \cap K_2$. The mapping $\delta_2 : \mathscr{F}(K_2) \to \mathscr{F}(K_1)$ is defined in a similar way. Then, for any $F, G \trianglelefteq K$, we have:*

(a) *If $F \trianglelefteq G$ then $\delta_1(G) \trianglelefteq \delta_1(F)$.*
(b) $\delta_1(F \wedge G) \trianglerighteq \delta_1(F) \vee \delta_1(G)$.
(c) $\delta_1(F \vee G) = \delta_1(F) \wedge \delta_1(G)$.
(d) $F \trianglelefteq \delta_2 \circ \delta_1(F)$.
(e) $\delta_1 \circ \delta_2 \circ \delta_1(F) = \delta_1(F)$.
(f) *The mapping $\delta_2 \circ \delta_1 : \mathscr{F}(K_1) \to \mathscr{F}(K_1)$ is a closure operation.*
(g) *$F = \delta_2(E)$ for some $E \trianglelefteq K_2$ if and only if $\delta_2 \circ \delta_1(F) = F$.*

We shall denote the closure operation $\delta_K^* \circ d_K$ by cl_K.

PROPOSITION 2.5. *Let $K_1, K_2, \delta_1, \delta_2$ have the same meanings as in Proposition 2.4.*

(a) *The following are equivalent statements:*
  (i) *δ_1 is injective;*
  (ii) *for any $F \trianglelefteq K$, $\delta_2 \circ \delta_1(F) = F$;*
  (iii) *δ_2 is surjective;*
  (iv) *for any $F, G \trianglelefteq K$, we have $G \trianglelefteq F$ iff $\delta_1(F) \trianglelefteq \delta_1(G)$.*
(b) *When δ_1 is bijective, we have, for any $F, G \trianglelefteq K$, $\delta_1(F \wedge G) = \delta_1(F) \vee \delta_1(G)$.*

COROLLARY 2.6. *For any cone K, d_K is injective iff each face of K is exposed.*

REMARK 2.7. The injectivity or surjectivity of the duality operator d_K of a cone K is an intrinsic property of K. It does not depend on the inner product of the underlying space (Tam [13, Corollary 1 of Theorem 1.4]) (indeed, it can be defined in the more general setting of a vector space and its dual space). Nor does it depend on the space containing it: let K^D be the dual cone of K in its own linear span, and denote by $\bar{d}_K$ the duality operator from $\mathscr{F}(K)$ to $\mathscr{F}(K^D)$. Then d_K is injective (surjective) iff $\bar{d}_K$ is injective (surjective).

PROPOSITION 2.8. *Let K be a cone which is the direct sum of a linear subspace L and a cone K_1. Then the lattices $\mathscr{F}(K)$ and $\mathscr{F}(K_1)$ are isomorphic. Further, d_K is injective (surjective) iff d_{K_1} is injective (surjective).*

According to Proposition 2.5(b), the condition "for any $F, G \trianglelefteq K$, $d_K(F \wedge G) = d_K(F) \vee d_K(G)$" is implied by the bijectivity of d_K. That the converse is false, even for proper cones, can be illustrated by the cone K_2 considered in Example 4.2 of Barker [2]. For a proper cone K, it is also known that the above condition implies the injectivity of d_K (see Tam [13, Corollary 4 of Theorem 1.3], and Corollary 2.13 below). However, for a nonclosed cone K, this may not be so, as can be illustrated by the following example.

EXAMPLE 2.9. Let M be the compact convex set in R^2 bounded by the ξ_1-axis and the curves $\xi_1 = 1 + \sqrt{\xi_2}$, $\xi_1 = -1 - \sqrt{\xi_2}$ and $\xi_1^2 + (\xi_2 - 2)^2 = 5$ (see Figure 1).

Let C be the convex set $M \sim \{(-1,0)\}$, and let K be the cone in R^3 given by $K = \{\alpha(\xi_1, \xi_2, 1): (\xi_1, \xi_2) \in C, \alpha \geqslant 0\}$. It is easily checked that d_K possesses the following property: for any $F, G \trianglelefteq K$, $d_K(F \wedge G) = d_K(F) \vee d_K(G)$. However, d_K is not injective, as $d_K[\varphi(1,0,0)] = d_K[\varphi(0,0,1)] =$

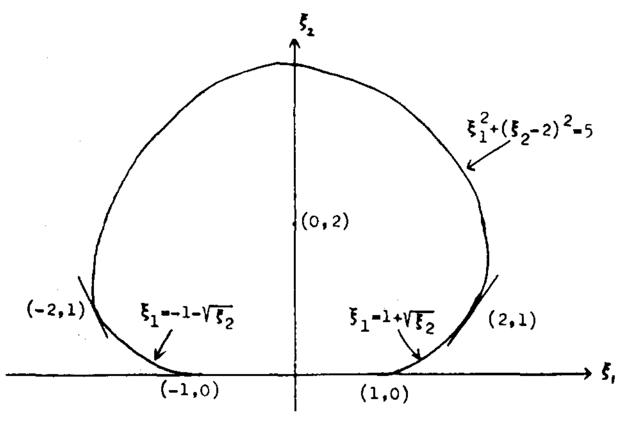

FIG. 1. The convex set M.

$\varphi(0,1,0)$ (face of K^*). Note also that the face lattice $\mathscr{F}(K)$ is complemented, but not section complemented.

LEMMA 2.10. *If the face lattice $\mathscr{F}(K)$ is complemented, then the lineality space of K is an exposed face of K.*

Proof. In view of Proposition 2.1(d) it suffices to show that no vector x of K satisfies $-x \in \operatorname{cl} K \sim K$. Let us assume the contrary, that x is such a vector. As $\mathscr{F}(K)$ is complemented, there exists a vector $y \in K$ such that $\varphi(x) \vee \varphi(y) = K$ and $\varphi(x) \wedge \varphi(y) = L$, where L is the lineality space of K. Then since $\varphi(x+y) = \varphi(x) \vee \varphi(y) = K$, we have, $x + y \in \operatorname{relint} K$. Further, as $-x \in \operatorname{cl} K$, we have $(x+y) + \alpha(-x) \in \operatorname{relint}(\operatorname{cl} K) = \operatorname{relint} K$ for all positive α. Hence $x \in \varphi(y)$, or $\varphi(x) \wedge \varphi(y) = \varphi(x)$. So $x \in L$, which is a contradiction. ∎

PROPOSITION 2.11. *Let K be a cone with a section complemented face lattice. If, for all $F, G \trianglelefteq K$, we have $d_K(F) \vee d_K(G) = K^*$ whenever $F \wedge G = L$ (the lineality space of K), then d_K is injective.*

Proof. Assume the contrary, that d_K is not injective. Then K has a nonexposed face F. In view of Lemma 2.10, $F \neq L$. Now F is a proper face of $\operatorname{cl}_K(F)$, and as $\mathscr{F}(K)$ is section complemented, there exists $G \trianglelefteq K$ such that $F \wedge G = L$ and $F \vee G = \operatorname{cl}_K(F)$. From $G \trianglelefteq \operatorname{cl}_K(F)$, we obtain $d_K(F) = d_K(\operatorname{cl}_K(F)) \trianglelefteq d_K(G)$; hence as $F \wedge G = L$, we have $d_K(G) = d_K(F) \vee d_K(G) = K^* = d_K(L)$. So $G \trianglelefteq \delta_K^* \circ d_K(G) = \delta_K^* \circ d_K(L) = L$, as L is an exposed face of K by Lemma 2.10. Thus $G = L$, and hence $\operatorname{cl}_K(F) = F \vee G = F$, which is a contradiction. ∎

COROLLARY 2.12. *Let K be a cone with a section complemented face lattice (for instance, when K is closed). If for all $F, G \trianglelefteq K$, we have $d_K(F \wedge G) = d_K(F) \vee d_K(G)$, then d_K is injective.*

PROPOSITION 2.13 (cf. Tam [13, Theorem 1.3(v)]). *Let K be a closed cone. Then d_K is injective iff for all $F, G \trianglelefteq K$, $F \wedge G = L$ implies $d_K(F) \vee d_K(G) = K^*$.*

Proof. The "if" part follows from Corollary 2.12. To prove the "only iff" part, let $F, G \trianglelefteq K$ such that $F \wedge G = L$. As d_K is injective, $F = \delta_K^* \circ d_K(F)$ and $G = \delta_K^* \circ d_K(G)$. Hence, $L = \delta_K^* \circ d_K(F) \wedge \delta_K^* \circ d_K(G) = \delta_K^*[d_K(F) \vee d_K(G)]$, so by Proposition 2.1(e), $d_K(F) \vee d_K(G) = K^*$. ∎

Again, in the "only if" part of Proposition 2.13, the closedness of K is a crucial condition: it cannot be replaced by, say, $\mathscr{F}(K)$ being section complemented as in Proposition 2.11. We give an example below.

EXAMPLE 2.14. *Let K be the following cone in R^3:*

$$\{(\xi_1, \xi_2, \xi_3) \in R^3 : \xi_i > 0\} \cup F_1 \cup F_2,$$

where

$$F_1 = \{(\xi_1, \xi_2, 0) : 0 < \xi_2 < \xi_1\} \cup \{(0,0,0)\}$$

and

$$F_2 = \{(0, \xi_2, \xi_3) : 0 < \xi_2 < \xi_3\} \cup \{(0,0,0)\}.$$

It is easily checked that K has exactly four faces, namely, 0, F_1, F_2, and K. Also $\mathscr{F}(K)$ is relatively complemented, and d_K is injective. Note, however, that $F_1 \wedge F_2 = 0$, but $d_K(F_1) \vee d_K(F_2) = \varphi(0,0,1) \vee \varphi(1,0,0) \neq R^3_+ = K^*$.

3. THE POINT'S CONE

Let C be a convex set in R^n. For any $y \in C$, the set $\{\alpha(x-y) : \alpha \geqslant 0$ and $x \in C\}$ is a cone, known as the *cone of C at y* and denoted by $\operatorname{cone}(y, C)$. When $y \in \operatorname{rbd} C$ (the relative boundary of C), the closure of $\operatorname{cone}(y, C)$ is often referred to as the *cone of support of C at y*, and the dual of $-\operatorname{cone}(y, C)$, the *normal cone of C at y* (see, for instance, Valentine [5, p. 135]). In [16] Waksman and Epelman have offered a classification of points of a finite dimensional convex set based on a function specially defined on the convex set as well as on the concept of the point's cone with respect to the set. We shall also make use of the concept of the point's cone in our study of the duality operator of a cone.

Hereafter we shall always use K to denote a cone in R^n.

PROPOSITION 3.1. *For any $y \in K$, we have* $\operatorname{cone}(y, K) = K + \operatorname{span} \varphi(y) = K - \varphi(y)$.

Proof. The proof is straightforward. The following two facts are used: $\operatorname{span} \varphi(y) = \varphi(y) - \varphi(y)$ and $\varphi(y) = \{x \in K : y - \alpha x \in K$ for some $\alpha > 0\}$. ■

For any neighborhood N of y, it is easily shown that $\mathrm{cone}(y, K \cap N) = \mathrm{cone}(y, K)$. So $\mathrm{cone}(y, K)$ depends only on the local behavior of K around y; in fact, it also determines the dual face of $\varphi(y)$.

COROLLARY 3.2. *For any* $y \in K$, *the dual cones of* $\mathrm{cone}(y, K)$ *and* $d_K(\varphi(y))$ *in* R^n *are respectively given by* $\mathrm{cone}(y, K)^* = d_K(\varphi(y))$ *and* $[d_K(\varphi(y))]^* = \mathrm{cl}[\mathrm{cone}(y, K)]$.

Proof. $\mathrm{cone}(y, K)^* = [K + \mathrm{span}\,\varphi(y)]^* = K^* \cap [\mathrm{span}\,\varphi(y)]^{\perp} = d_K(\varphi(y))$. $[d_K(\varphi(y))]^* = \mathrm{cone}(y, K)^{**} = \mathrm{cl}[\mathrm{cone}(y, K)]$.

We digress at this point to give an alternative proof of the following known result (see Schneider and Vidyasagar [11]).

COROLLARY 3.3. *Let K be a proper cone in R^n. Denote by $\Sigma(K)$ the set of all matrices cross-positive on K, and by $\pi_1(K)$ the set $\{A + \alpha I : A \in \pi(K)$ and $\alpha \in R\}$, where $\pi(K) = \{A : AK \subseteq K\}$. Then $\Sigma(K) = cl[\pi_1(K)]$.*

Proof. Recall that an $n \times n$ real matrix $A \in \Sigma(K)$ iff for all $y \in K$ and $z \in K^*$, we have $(z, Ay) \geqslant 0$ whenever $(z, y) = 0$. On the space of $n \times n$ real matrices we introduce the inner product $(B, A) = \mathrm{trace}\, B^T A$. Then this inner product and the usual inner product of R^n are related by $(zy^T, A) = (z, Ay)$. Thus, $A \in \Sigma(K)$ iff $(zy^T, A) \geqslant 0$ whenever $(zy^T, I) = 0$ where $y \in K$ and $z \in K^*$. Now, the dual cone of $\pi(K)$ is equal to the positive hull of all matrices of the form zy^T with $y \in K$ and $z \in K^*$ (Tam [12, Theorem 1]). Consequently, $\Sigma(K) = [d_{\pi(K)}(\varphi(I))]^*$. And so by Corollary 3.2, $\Sigma(K) = \mathrm{cl}[\mathrm{cone}(I, \pi(K))] = \mathrm{cl}[\pi_1(K)]$. ∎

PROPOSITION 3.4. *For any $y \in K$, we have*

$$\varphi(y) = -\mathrm{cone}(y, K) \cap K \quad \textit{and} \quad \mathrm{cl}_K(\varphi(y)) = -\mathrm{cl}[\mathrm{cone}(y, K)] \cap K.$$

Proof. Since $-\varphi(y) \subseteq \mathrm{span}\,\varphi(y) \subseteq \mathrm{cone}(y, K)$, we have $\varphi(y) \subseteq -\mathrm{cone}(y, K)$. Hence $\varphi(y) \subseteq -\mathrm{cone}(y, K) \cap K$. On the other hand, $K \cap (-\mathrm{cone}(y, K)) = K \cap [-K + \varphi(y)] \subseteq K \cap \varphi(y) = \varphi(y)$. Therefore, $\varphi(y) = -\mathrm{cone}(y, K) \cap K$.

Since $-\mathrm{cone}(y, K) \cap K$ is orthogonal to the set $d_K(\varphi(y))$, so is $-\mathrm{cl}[\mathrm{cone}(y, K)] \cap K$. Hence, $-\mathrm{cl}[\mathrm{cone}(y, K)] \cap K \subseteq \mathrm{cl}_K(\varphi(y))$. Conversely, if $x \in \mathrm{cl}_K(\varphi(y))$ then $x \in K$ and $(x, d_K(\varphi(y))) = 0$. Hence, $x \in$

$-[d_K(\varphi(y))]^* \cap K = -\operatorname{cl}[\operatorname{cone}(y, K)] \cap K$. We have shown that $\operatorname{cl}_K(\varphi(y)) = -\operatorname{cl}[\operatorname{cone}(y, K)] \cap K$. ■

COROLLARY 3.5. *For any $F \trianglelefteq K$, we have $F = K \cap (F - K)$ and $\operatorname{cl}_K(F) = F \cap \operatorname{cl}(F - K)$.*

Proof. Let $y \in \operatorname{relint} F$. Then $F = \varphi(y)$. By proposition 3.4, $\varphi(y) = -\operatorname{cone}(y, K) \cap K$, and by Proposition 3.1, $-\operatorname{cone}(y, K) = \varphi(y) - K = F - K$. Thus, $F = K \cap (F - K)$. Similarly, we can prove $\operatorname{cl}_K(F) = K \cap \operatorname{cl}(F - K)$. ■

COROLLARY 3.6. *If $x, y \in K$, then*

(a) $\varphi(x) \trianglelefteq \varphi(y)$ *iff* $\operatorname{cone}(x, K) \subset \operatorname{cone}(y, K)$ (*cf.* [16, *Lemma* 9(a)]),
(b) $\operatorname{cl}_K(\varphi(x)) \trianglelefteq \operatorname{cl}_K(\varphi(y))$ *iff* $\operatorname{cl}[\operatorname{cone}(x, K)] \subseteq \operatorname{cl}[\operatorname{cone}(y, K)]$.

Proof. (a) follows readily from Propositions 3.1 and 3.4.
(b): $\operatorname{cl}_K(\varphi(x)) \trianglelefteq \operatorname{cl}_K(\varphi(y))$

iff $\delta_K^* \circ d_K(\varphi(x)) \trianglelefteq \delta_K^* \circ d_K(\varphi(y))$
iff $d_K(\varphi(y)) \trianglelefteq d_K(\varphi(x))$
iff $[d_K(\varphi(x))]^* \trianglelefteq [d_K(\varphi(y))]^*$ [we need the closedness of $d_K(\varphi(x))$ and $d_K(\varphi(y))$ for the "if" part]
iff $\operatorname{cl}[\operatorname{cone}(x, K)] \subseteq \operatorname{cl}[\operatorname{cone}(y, K)]$. ■

COROLLARY 3.7. *For any $x \in K$ we have $x \in L$ iff* $\operatorname{cone}(x, K) = K$, *where L is the lineality space of K.*

Proof. By Corollary 3.6(a), we have $\operatorname{cone}(x, K) = K$ iff $\operatorname{cone}(x, K) = \operatorname{cone}(0, K)$ iff $\varphi(x) = \varphi(0)$ iff $x \in L$. ■

THEOREM 3.8. *For any vector $y \in K$, the mapping $\eta: [\varphi(y), K] \to \mathscr{F}[\operatorname{cone}(y, K)]$ given by $\eta(F) = \operatorname{cone}(y, F)$ is a lattice isomorphism, where $[\varphi(y), K]$ is the interval sublattice $\{F \trianglelefteq K : \varphi(y) \trianglelefteq F\}$ of $\mathscr{F}(K)$.*

Proof. We first show that if $F \trianglelefteq K$ such that $\varphi(y) \trianglelefteq F$ then $\operatorname{cone}(y, F) \trianglelefteq \operatorname{cone}(y, K)$. Clearly, $\operatorname{cone}(y, F)$ is a subcone of $\operatorname{cone}(y, K)$. Let $w \in \operatorname{cone}(y, F)$ such that $w = w_1 + w_2$ for some $w_1, w_2 \in \operatorname{cone}(y, K)$. Then there exist $x \in F$, $x_1, x_2 \in K$, and $\alpha, \alpha_1, \alpha_2 \geqslant 0$ such that $w = \alpha(x - y)$ and $w_i = \alpha_i(x_i - y)$, $i = 1,2$. It is required to show that $w_1, w_2 \in \operatorname{cone}(y, F)$.

If $\alpha = 0$, we have $\alpha_1(x_1 - y) + \alpha_2(x_2 - y) = 0$, or $\alpha_1 x_1 + \alpha_2 x_2 = (\alpha_1 + \alpha_2)y \in F$. Hence $\alpha_1 x_1, \alpha_2 x_2 \in F$. If $\alpha_1 \neq 0$, then $x_1 \in F$ and hence $w_1 \in \text{cone}(y, F)$. If $\alpha_1 = 0$, then clearly w_1 is the zero vector and belongs to $\text{cone}(y, F)$. Similarly we can show that $w_2 \in \text{cone}(y, F)$.

If $\alpha \neq 0$, we may without loss of generality assume $\alpha = 1$. We shall consider the case $\alpha_1 + \alpha_2 > 1$. The case $\alpha_1 + \alpha_2 \leqslant 1$ can be treated similarly. Then we have $x + (\alpha_1 + \alpha_2 - 1)y = \alpha_1 x_1 + \alpha_2 x_2$. As $x, y \in F$, the vector on the left side belongs to F. Thus, $\alpha_1 x_1, \alpha_2 x_2 \in F$, from which we can deduce $w_1, w_2 \in \text{cone}(y, F)$.

So η is a well-defined mapping. Now let $M \trianglelefteq \text{cone}(y, K)$. Let $F = \{x \in K : x - y \in M\}$. Clearly $y \in F$. We contend that $F \trianglelefteq K$. Let $x_1, x_2 \in K$ such that $x_1 + x_2 \in F$. Then $x_1 + x_2 - y \in M$. Note that $-y$ belongs to the lineality space of $\text{cone}(y, K)$, as $\text{span}\,\varphi(y) \subseteq \text{cone}(y, K)$. But every face of $\text{cone}(y, K)$ contains the lineality space of $\text{cone}(y, K)$, whence $-y \in M$. Thus $(x_1 - y) + (x_2 - y) = (x_1 + x_2 - y) + (-y) \in M$, the right side being the sum of two vectors in M. As M is a face of $\text{cone}(y, K)$, this implies $x_1 - y, x_2 - y \in M$. Consequently, $x_1, x_2 \in F$. In a similar way, we can show that F is itself a cone. Therefore, F is a face of K. From the definition of F, it is also clear that $\text{cone}(y, F) = M$. We have thus established the surjectivity of η.

Let $F_1, F_2 \trianglelefteq K$, both containing the vector y. We are going to show that if $F_1 \not\subseteq F_2$ then $\text{cone}(y, F_1) \not\subseteq \text{cone}(y, F_2)$. Assume that the contrary holds. Then $F_1 \subseteq F_1 + \text{span}\,\varphi(y) = \text{cone}(y, F_1) \subseteq \text{cone}(y, F_2) = F_2 + \text{span}\,\varphi(y)$, whence $F_1 = \text{span}\, F_1 \cap K \subseteq \text{span}[F_2 + \text{span}\,\varphi(y)] \cap K = \text{span}\, F_2 \cap K = F_2$, which is a contradiction. The injectivity of η now clearly follows. We also have $F_1 \trianglelefteq F_2$ iff $\text{cone}(y, F_1) \trianglelefteq \text{cone}(y, F_2)$.

We have proved that η is an isomorphism between $[\varphi(y), K]$ and $\mathscr{F}(K)$ when considered as partially ordered sets, and hence an isomorphism between these sets when considered as lattices. ■

Corollary 3.9. *The lineality space of* $\text{cone}(y, K)$ *is* $\text{span}\,\varphi(y)$.

Proof. Note that $\text{span}\,\varphi(y) = \text{cone}(y, \varphi(y))$, since $y \in \text{relint}\,\varphi(y)$. As $\varphi(y)$ is a face of K containing y, $\text{cone}(y, \varphi(y)) \trianglelefteq \text{cone}(y, K)$. On the other hand, as $\text{span}\,\varphi(y)$ is a linear subspace contained in $\text{cone}(y, K)$, necessarily it is contained in the lineality space of cone (y, K). Hence, $\text{span}\,\varphi(y)$ is the lineality space of $\text{cone}(y, K)$. ■

Corollary 3.10. *Let F_1, F_2 be faces of K, both containing the vector y. Then*

(a) $\text{cone}(y, F_1 \wedge F_2) = \text{cone}(y, F_1) \wedge \text{cone}(y, F_2)$,
(b) $\text{cone}(y, F_1 \vee F_2) = \text{cone}(y, F_1) \vee \text{cone}(y, F_2)$.

REMARK 3.11. Another way of looking at cone(y, K) is using the concept of quotient cones. If K is a cone in R^n and $y \in K$, we can construct the quotient space $R^n/\operatorname{span}\varphi(y)$. It is not difficult to show that in this quotient space the set $K/\operatorname{span}\varphi(y)$ forms a pointed cone. Furthermore, the face lattices of the cones $K/\operatorname{span}\varphi(y)$ and cone(y, K) are isomorphic, because we have $M \trianglelefteq K/\operatorname{span}\varphi(y)$ iff $M = F/\operatorname{span}\varphi(y)$ for some $F \trianglelefteq K$ such that $y \in F$.

4. THE DUALITY OPERATOR OF THE POINT'S CONE

For each $y \in K$, we shall denote, for simplicity, $d_{\operatorname{cone}(y,K)}$ by d_y and $\delta^*_{\operatorname{cone}(y,K)}$ by δ^*_y. Note that d_y is from $\mathscr{F}[\operatorname{cone}(y,K)]$ to $\mathscr{F}[d_K(\varphi(y))]$. We shall show that the properties of d_K, the duality operator of K, are reflected from those of d_y. It is known that for each proper cone K, which is nonpolyhedral, there always exists a vector $y \in K$ such that cone(y, K) is not closed (see Waksman and Epelman [16, §8 and Theorem 10]). So the duality operators of nonclosed cones are naturally involved in our investigation.

PROPOSITION 4.1.

(a) *If* $F \trianglelefteq$ *is such that* $\varphi(y) \trianglelefteq F$, *then* $d_y[\operatorname{cone}(y,F)] = d_K(F)$.
(b) *If* $G \trianglelefteq d_K(\varphi(y))$ *then* $\delta^*_y(G) = \operatorname{cone}(y, \delta^*_K(G))$.

Proof. (a): $d_y[\operatorname{cone}(y,F)] = [\operatorname{span}\operatorname{cone}(y,F)]^{\perp} \cap d_K(\varphi(y)) = (\operatorname{span} F)^{\perp} \cap [K^* \cap d_K(\varphi(y))] = [(\operatorname{span} F)^{\perp} \cap K^*] \cap d_K(\varphi(y)) = d_K(F)$, the last equality being true because $\varphi(y) \trianglelefteq F$ implies $d_K(F) \trianglelefteq d_K(\varphi(y))$.

(b): Let $G \trianglelefteq d_K(\varphi(y))$. Then $\delta^*_K(G) \trianglerighteq \delta^*_K \circ d_K(\varphi(y)) \trianglerighteq \varphi(y)$, so $\operatorname{cone}(y, \delta^*_K(G)) \trianglelefteq \operatorname{cone}(y,K)$. Let $x \in \delta^*_K(G)$. Then $(x-y, G) = (x,G) - (y,G) = 0$. Hence $\operatorname{cone}(y, \delta^*_K(G)) \subseteq \delta^*_y(G)$. Now let $u \in \delta^*_y(G)$. Then $u = \alpha(x-y)$ for some $x \in K$ and $\alpha > 0$. From $(u,G) = 0 = (y,G)$, we obtain $(x,G) = 0$; hence $x \in \delta^*_K(G)$. Therefore, $u \in \operatorname{cone}(y, \delta^*_K(G))$. The proof is complete. ∎

COROLLARY 4.2. *For any* $F \trianglelefteq K$ *satisfying* $y \in F$, *we have*

$$\operatorname{cone}(y,F) \textit{ is an exposed face of } \operatorname{cone}(y,K)$$

iff F *is an exposed face of* K.

Proof. cone(y, F) is an exposed face of cone(y, K)
iff $\delta_y^* \circ d_y[\text{cone}(y, F)] = \text{cone}(y, F)$
iff $\text{cone}(y, \delta_K^* \circ d_K(F)) = \text{cone}(y, F)$
iff $\delta_K^* \circ d_K(F) = F$
iff F is an exposed face of K.

COROLLARY 4.3. For any vector $y \in K$, the duality operator d_y of cone(y, K) is injective iff each face of K which contains $\varphi(y)$ is an exposed face of K.

Proof. By Corollary 2.6, d_y is injective iff each face of cone(y, K) is exposed. Now each face of cone(y, K) is of the form cone(y, F) for some $\varphi(y) \trianglelefteq F \trianglelefteq K$. So, by Corollary 4.2, our result follows. ■

REMARK 4.4. Putting the results of Proposition 4.1 differently, we obtain the following commutative diagrams:

$$\begin{array}{ccc} d_K(\varphi(y)) & \xrightarrow{i} & \mathscr{F}(K^*) \\ \uparrow{\scriptstyle d_y} & & \uparrow{\scriptstyle d_K} \\ \mathscr{F}[\text{cone}(y, K)] & \xrightarrow{\eta^{-1}} & \mathscr{F}(K) \end{array} \qquad \begin{array}{ccc} d_K(\varphi(y)) & \xrightarrow{i} & \mathscr{F}(K^*) \\ \downarrow{\scriptstyle \delta_y^*} & & \downarrow{\scriptstyle \delta_K^*} \\ \mathscr{F}[\text{cone}(y, K)] & \xrightarrow{\eta^{-1}} & \mathscr{F}(K) \end{array}$$

where i is the inclusion map and η^{-1} is the mapping given by $\eta^{-1}[\text{cone}(y, F)] = F$.

THEOREM 4.5. *Let K be a cone in R^n such that its lineality space L is an exposed face of K. The following statements are equivalent:*

(i) *d_K is injective.*
(ii) *For each $y \in K \sim L$, the duality operator d_y is injective.*

If, in addition, K is closed and pointed, the following is also another equivalent condition:

(iii) *For each nonzero extreme vector $y \in K$, d_y is injective.*

Proof. (i) ⇒ (ii): If d_K is injective, then every face of K is exposed. So, by Corollary 4.3, (ii) holds.

(ii) ⇒ (i): It suffices to show that each face of K is exposed. By assumption, the lineality space L of K is exposed. So consider a face F of K, different from L. Choose a vector $y \in F \sim L$. By (ii), d_y is injective. Hence, the face cone(y, F) of cone(y, K) is exposed. So by Corollary 4.2, F is exposed.

Now suppose that K is closed and pointed. Then the lineality space of K is just $\{0\}$, and each face of K is generated by its extreme vectors. Clearly then (ii) $\Rightarrow$ (iii). In the above proof of (ii) $\Rightarrow$ (i), if we require y to be a nonzero extreme vector of F, we obtain a proof of (iii) $\Rightarrow$ (i). ■

Note that in condition (ii) of Theorem 4.5 above, we deliberately omit the case $y \in L$. Indeed, if $y \in L$, then $\varphi(y) = \varphi(0)$ and so by Corollary 3.6(a) cone$(y, K) =$ cone$(0, K) = K$. Hence $d_y = d_K$. Note also that in this theorem the exposedness assumption on L is crucial. As an example, one may consider the cone $K = \{(\xi_1, \xi_2) \in R^2 : \xi_2 \geqslant 0\} \sim \{(\xi_1, 0) : \xi_1 < 0\}$. The following can be checked: K is a pointed cone, and condition (ii) of Theorem 4.6 is satisfied. However, d_K is not injective, as $\{0\}$ is not an exposed face of K.

The following result is related to Theorem 4.5.

PROPOSITION 4.6. *Let $y \in K$. The following statements are equivalent:*

(i) *$\varphi(y)$ is a nonexposed face of K.*

(ii) *There exists a vector $w \in$ cone(y, K) such that $-w \in$ cl[cone(y, K)] $\sim$ cone(y, K).*

Proof. By Proposition 2.1(d), condition (ii) is equivalent to the lineality space of cone(y, K), or cone$(y, \varphi(y))$, being a nonexposed face of cone(y, K). By Corollary 4.2, cone$(y, \varphi(y))$ is an exposed face of cone(y, K) iff $\varphi(y)$ is an exposed face of K. So the equivalence of (i) and (ii) follows. ■

For the surjectivity of the duality operators we have the following analogous result, whose proof we leave to the reader.

THEOREM 4.7. *Let K be a closed cone in R^n. The following statements are equivalent:*

(i) *d_K is surjective.*

(ii) *For each $y \in K \sim L$, the duality operator d_y is surjective.*

If in addition K is pointed, then the following is also an equivalent condition:

(iii) *For each nonzero extreme vector y of K, d_y is surjective.*

Again in Theorem 4.7 the closedness assumption on K is not redundant. For example, if K is the cone$\{(\xi_1, \xi_2) \in R^2 : \xi_1 \geqslant 0$ and $\xi_2 > 0\} \cup \{(0,0)\}$, then for any nonzero $y \in K$, cone(y, K) is either R^2 or a closed half space of R^2, so that condition (ii) of Theorem 4.7 is satisfied. However, d_K is not surjective: K does not have a face whose dual face is the face $\varphi(0,1)$ of K^*.

The following result is known and is not difficult to prove.

PROPOSITION 4.8. *Let C be a convex set in R^n, and let $K = \{\alpha(x,1) \in R^{n+1} : \alpha \geq 0$ and $x \in C\}$. Then K is a cone in R^{n+1}. Further, the face lattices $\mathscr{F}(C)$ and $\mathscr{F}(K)$ are isomorphic under the isomorphism $F \mapsto \tilde{F}$, where F is a face of C and $\tilde{F} = \{\alpha(x,1) : \alpha \geq 0$ and $x \in F\}$; moreover, in this isomorphism, exposed faces of C correspond to exposed faces of K.*

(For the definitions of faces and exposed faces of a convex set, see for instance Rockafellar [10].)

If K and C are given as in Proposition 4.8, their points' cones are also related in a simple way.

PROPOSITION 4.9. *Let K and C have the same meanings as in Proposition 4.8. For any $y \in C$, we have*

$$\operatorname{cone}((y,1), K) = [\operatorname{cone}(y,C),0] \oplus \operatorname{span}\{(y,1)\},$$

where

$$[\operatorname{cone}(y,C),0] = \{(x,0) \in R^{n+1} : x \in \operatorname{cone}(y,C)\}.$$

(Cf. Waksman and Epelman [16, Lemma 9(b)].)

Many of the results in Sections 3 and 4 about the point's cone of a cone are also true (or have analogues) for the points' cones of a convex set. For completeness, some of them are collected below. They can be either proved directly or deduced from the corresponding results for the point's cone of a cone.

PROPOSITION 4.10. *If y belongs to a convex set C in R^n, then $\varphi(y) = [\{y\} - \operatorname{cone}(y,C)] \cap C$ and $\operatorname{cl}_C(\varphi(y)) = [\{y\} - \operatorname{cl}(\operatorname{cone}(y,C))] \cap C$, where $\varphi(y)$ is the face of C generated by y, and $\operatorname{cl}_C(\varphi(y))$ is the intersection of all exposed faces of C which contain $\varphi(y)$ (or equivalently, the intersection of all supporting hyperplanes of C which contain $\varphi(y)$).*

PROPOSITION 4.11. *Let C be a convex set in R^n, and let $y \in C$. Then the mapping $\eta : [\varphi(y), C] \to \mathscr{F}[\operatorname{cone}(y,C)]$ given by $\eta(F) = \operatorname{cone}(y, F)$ is a lattice isomorphism.*

PROPOSITION 4.12. *Let $y \in C$, where C is a convex set in R^n. The following statements are equivalent*:

(i) *$\varphi(y)$ is a nonexposed face of C.*
(ii) *There exists a vector $w \in \operatorname{cone}(y, C)$ such that $-w \in \operatorname{cl}[\operatorname{cone}(y, C)] \sim \operatorname{cone}(y, C)$.*

COROLLARY 4.13. *Let C_1, C_2 be convex sets in R^n, and let $y \in C_1 \cap C_2$. If the faces generated by y in C_1 and in C_2 are exposed, then the face of $C_1 \cap C_2$ generated by y is also exposed.*

Proof. Suppose that the face of $C_1 \cap C_2$ generated by y is nonexposed. Then by Proposition 4.12, there exists a vector $w \in \operatorname{cone}(y, C_1 \cap C_2)$ such that $-w \in \operatorname{cl}[\operatorname{cone}(y, C_1 \cap C_2)] \sim \operatorname{cone}(y, C_1 \cap C_2)$. Now by Proposition 5.1(a), $\operatorname{cone}(y, C_1 \cap C_2) = \operatorname{cone}(y, C_1) \cap \operatorname{cone}(y, C_2)$, so $w \in \operatorname{cone}(y, C_1) \cap \operatorname{cone}(y, C_2)$ and $-w \in \operatorname{cl}[\operatorname{cone}(y, C_1)] \cap \operatorname{cl}[\operatorname{cone}(y, C_2)]$; further, $-w$ does not belong to either $\operatorname{cone}(y, C_1)$ or $\operatorname{cone}(y, C_2)$. Thus, for $i = 1$ or 2, we have $w \in \operatorname{cone}(y, C_i)$ but $-w \in \operatorname{cl}[\operatorname{cone}(y, C_i)] \sim \operatorname{cone}(y, C_i)$. So, by Proposition 4.12, the face generated by y in C_1 or in C_2 is nonexposed, which is a contradiction. ∎

COROLLARY 4.14. *If K_1, K_2 are cones in R^n such that both d_{K_1} and d_{K_2} are injective, then $d_{K_1 \cap K_2}$ is also injective.*

Note, however, that the corresponding result for the surjectivity of $d_{K_1 \cap K_2}$ does not hold. As an example, let K_1 and K_2 be cones in R^3 given by $K_i = \{\alpha(x, 1): \alpha \geqslant 0 \text{ and } x \in C_i\}$, where C_1 and C_2 are the compact convex sets in R^2 bounded respectively by the circles with equations $(\xi_1 - 1)^2 + \xi_2^2 = 2$ and $(\xi_1 + 1)^2 + \xi_2^2 = 2$. Then d_{K_1} and d_{K_2} are both bijective, but $d_{K_1 \cap K_2}$ is not surjective, as the duality operator of $\operatorname{cone}((0,1), C_1 \cap C_2)$ is not surjective.

EXAMPLE 4.15. Let K be the cone in R^4 given by $K = \{\alpha(x, 1): \alpha \geqslant 0 \text{ and } x \in C\}$, where C is the convex set in R^3 as shown in Figure 2. (The cone K has already appeared in Example 3.5 of Barker [2].)

Note that all of the extreme points of the convex set C are exposed; hence so are the extreme rays of the cone K. However, F_4 is a nonexposed face of C, and the corresponding face of K is also nonexposed. Thus the duality operator d_K of K is not injective. In fact, we can also show that d_K is not surjective. Consider the cone of C at y (see Figure 3). Note that $\operatorname{cl}[\operatorname{cone}(y, K)]$ is the 3-dimensional polyhedral cone generated by the vectors x_1, x_2, and x_4, and that $\operatorname{cone}(y, C) = \operatorname{cl}[\operatorname{cone}(y, C)] \sim \{\alpha x_4 + \beta x_3: \alpha > 0,\ \beta \geqslant 0\}$. There is a 2-dimensional face G of $[\operatorname{cone}(y, K)]^*$ which is orthogonal to the vector x_4 of

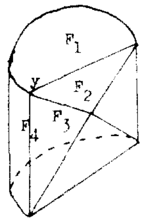

FIG. 2. The convex set C. The face F_3 is tangent to the cylindrical surface in F_4.

cl[cone(y, K)]. However, G is not the dual face of any face of cone(y, C), i.e. the duality operator of cone(y, C) is not surjective. So by Proposition 2.8 and 4.9, d_y, and hence d_K, is not surjective. In fact, d_y is also not injective, because $\varphi(x_3)$ and $\varphi(x_2 + x_3)$ are distinct faces of cone(y, K), which have the same dual face.

The following result is also not difficult to prove.

PROPOSITION 4.16. *Let K be a closed cone in R^n. The following are equivalent conditions:*

(i) *For any $F_1, F_2 \trianglelefteq K$, $d_K(F_1 \wedge F_2) = d_K(F_1) \vee d_K(F_2)$.*

(ii) *d_K is injective, and for each $y \in K \sim L$, the duality operator d_y satisfies the following: for all $G_1, G_2 \trianglelefteq$ cone(y, K), $d_y(G_1 \wedge G_2) = d_y(G_1) \vee d_y(G_2)$.*

If, in addition, K is pointed, then in condition (ii) *the phrase "for each $y \in K \sim L$" can be replaced by "for each nonzero extreme vector y of K."*

Again, in Theorem 4.16(ii) the condition that d_K is injective cannot be dropped. As an example, consider the cone $K = \{\alpha(x,1) : \alpha \geq 0, x \in M\}$,

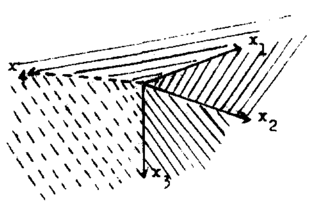

FIG. 3. cone(y, C).

where M is the compact convex set mentioned in Example 2.9. The following can be checked: for each nonzero $y \in K$, we have $d_y(G_1 \wedge G_2) = d_y(G_1) \vee d_y(G_2)$ for all $G_1, G_2 \trianglelefteq \text{cone}(y, K)$; d_K is not injective, and the condition "for all $F, G \trianglelefteq K$, $d_K(F \wedge G) = d_K(F) \vee d_K(G)$" is not satisfied.

For completeness, we include the following.

PROPOSITION 4.17. *For any cone K in R^n, the following conditions are equivalent*:

(i) *For any $F_1, F_2 \trianglelefteq K$, $d_K(F_1 \wedge F_2) = d_K(F_1) \vee d_K(F_2)$.*

(ii) *For any $G \trianglelefteq K^*$, there exists a smallest face F of K such that $d_K(F) \trianglelefteq G$.*

If, in addition, $\mathscr{F}(K)$ is section complemented, then the following is also an equivalent condition:

(iii) *d_K is injective, and the set of all exposed faces of K^* forms a sublattice of $\mathscr{F}(K^*)$.*

5. COMPACT CONVEX SETS WITH A PRESCRIBED POINT'S CONE AT THE ORIGIN

In this section we are concerned with the problem of constructing a compact convex set C whose point's cone at some given point y is equal to a prescribed cone. In view of $\text{cone}(y, C) = \text{cone}(0, C - \{y\})$, the given point y may be taken to be the origin of the space. The requirement that the constructed convex set C be closed is necessary; for if a cone K is given, we can always find a bounded convex set whose point's cone at the origin is equal to K, namely, $K \cap U$, where U is the unit ball centered at the origin. The following elementary result will be found useful.

PROPOSITION 5.1. *Let $C_1, \ldots, C_k$ be convex sets in R^n, and let $y \in \bigcap_{i=1}^k C_i$. Then*

(a) $\text{cone}(y, C_1 \cap \cdots \cap C_k) = \text{cone}(y, C_1) \cap \cdots \cap \text{cone}(y, C_k)$,

(b) $\text{cone}(y, \text{conv}(\bigcup_{i=1}^k C_i)) = \text{cone}(y, C_1) + \cdots + \text{cone}(y, C_k)$, *where* $\text{conv}(\bigcup_{i=1}^k C_i)$ *is the convex hull of the set* $\bigcup_{i=1}^k C_i$.

Proof. Clearly $\text{cone}(y, \bigcap_{i=1}^k C_i) \subseteq \bigcap_{i=1}^k \text{cone}(y, C_i)$. To prove the reverse inclusion, let $w \in \bigcap_{i=1}^k \text{cone}(y, C_i)$. Then, for each i, there exists $\alpha_i > 0$ such that $y + \alpha_i w \in C_i$. Let $\alpha = \min_i \alpha_i$. Then as $C_1, \ldots, C_k$ are convex, we have, $y + \alpha w \in \bigcap_{i=1}^k C_i$. Consequently, $w \in \bigcap_{i=1}^k \text{cone}(y, C_i)$. This proves (a).

To prove (b), let $x \in \operatorname{conv}(\bigcup_{i=1}^{k} C_i)$. Then $x = \alpha_1 x_1 + \cdots + \alpha_k x_k$ for some $x_i \in C_i$, $\alpha_i \geqslant 0$ with $\sum_{i=1}^{k} \alpha_i = 1$. Hence, $x - y = \alpha_1(x_1 - y) + \cdots + \alpha_k(x_k - y) \in \operatorname{cone}(y, C_1) + \cdots + \operatorname{cone}(y, C_k)$. Thus, $\operatorname{cone}(y, \operatorname{conv}(\bigcup_{i=1}^{k} C_i)) \subseteq \operatorname{cone}(y, C_1) + \cdots + \operatorname{cone}(y, C_k)$. The reverse inclusion is clearly satisfied. So (b) is proved. ■

REMARK 5.2. If $(C_i)_{i \in I}$ is an infinite sequence of convex sets in R^n such that $y \in \bigcap_{i \in I} C_i$, then we still have

$$\operatorname{cone}\left(y, \operatorname{conv}\left(\bigcup_{i \in I} C_i\right)\right) = \operatorname{cone}(y, C_1) + \operatorname{cone}(y, C_2) + \cdots,$$

provided that the right side is interpreted as the set of all possible finite sums of vectors taken from $\operatorname{cone}(y, C_1)$, $\operatorname{cone}(y, C_2), \ldots$. Note, however, that in this this case the set $\operatorname{conv}(\bigcup_{i \in I} C_i)$ may not be closed even when each C_i is compact.

PROPOSITION 5.3. *If K is a cone in R^n, then there exists a compact convex set C in R^n such that $0 \in C$ and* $\operatorname{cone}(0, C) = \operatorname{relint} K \cup \{0\}$.

Proof. Denote by K^D the dual of K in its own linear span. Let $T = \{z \in K^D : \|z\| = 1\}$. For each vector $z \in T$, let B_z be the closed unit ball of $\operatorname{span} K$ centered at z, i.e. $B_z = \{y \in \operatorname{span} K : \|y - z\| \leqslant 1\}$. Let $C = \bigcap_{z \in T} B_z$. Clearly, C is a compact convex set and $0 \in C$. We contend that $\operatorname{cone}(0, C) = \operatorname{relint} K \cup \{0\}$.

First, note that, if $x (\neq 0) \in \operatorname{span} K$ and $\alpha > 0$, then

$$\begin{aligned} \alpha x \in B_z \quad &\text{iff} \quad \|\alpha x - z\|^2 \leqslant 1 \\ &\text{iff} \quad \alpha^2 \|x\|^2 - 2\alpha(x, z) + \|z\|^2 \leqslant 1 \\ &\text{iff} \quad \alpha \leqslant 2(x, z)/\|x\|^2. \end{aligned}$$

It is known that $\operatorname{relint} K = \{y \in \operatorname{span} K : (z, y) > 0 \text{ for all nonzero } z \in K^D\}$ (cf. Schneider and Vidyasagar [11, Lemma 3]). Thus, for any $y \in \operatorname{span} K$, if $y \notin \operatorname{relint} K \cup \{0\}$, then there exists $z \in T$ such that $(y, z) \leqslant 0$. Hence, $\alpha y \notin B_z$ for all $\alpha > 0$, or $y \notin \operatorname{cone}(0, B_z)$. As $C \subseteq B_z$, necessarily, $\operatorname{cone}(0, C) \subseteq \operatorname{cone}(0, B_z)$, so $y \notin \operatorname{cone}(0, C)$. We have thus shown that $\operatorname{cone}(0, C) \subseteq \operatorname{relint} K \cup \{0\}$.

To prove the reverse inclusion, let $x \in \operatorname{relint} K$. Note that the real-valued function $z \mapsto (2x/\|x\|^2, z)$ is continuous. Denote by α its minimum on the compact set T. Then $\alpha > 0$. Furthermore, for any $z \in T$, we have, $\alpha \leqslant 2(x/\|x\|^2, z)$, so that $\alpha x \in B_z$; hence, $\alpha x \in \bigcap_{z \in T} B_z$, or $x \in \operatorname{cone}(0, C)$ as required. ■

It may be interesting to note that in the above proof of Proposition 5.3, if we had set $C = \bigcap_{z \in T_1} B_z$ or $\bigcap_{z \in T_2} B_z$, where $T_1 = \{ z \in \operatorname{rbd} K^D : \|z\| = 1\}$ and $T_2 = \{ z \in \operatorname{relint} K^D : \|z\| = 1\}$, we still could have obtained $\operatorname{cone}(0, C) = \operatorname{relint} K \cup \{0\}$. Note also that $\bigcap_{z \in T_1} \operatorname{cone}(0, B_z) = \bigcap_{z \in T} \operatorname{cone}(0, B_z) = \operatorname{relint} K \cup \{0\}$. However, $\bigcap_{z \in T_2} \operatorname{cone}(0, B_z) = \operatorname{cl} K \sim M$, where M is the lineality space of $\operatorname{cl} K$. This shows that if the number of convex sets C_i is infinite, Proposition 5.1(a) may or may not hold.

Proposition 5.4. *If K is a closed cone in R^n, then for any $F \trianglelefteq K$, there exists a compact convex set C in R^n, which contains the zero vector, such that* $\operatorname{cone}(0, C) = (K \sim F) \cup \{0\}$.

Proof. We first consider the case when F is an exposed face of K. Then there exists a vector $z \in K^*$, $\|z\| = 1$, such that $\delta_K^*[\varphi(z)] = F$. Let $C = K \cap B_z$, where $B_z = \{ y \in R^n : \|y - z\| \leqslant 1\}$. Then C is a compact convex set containing 0. Furthermore, $\operatorname{cone}(0, C) = \operatorname{cone}(0, K) \cap \operatorname{cone}(0, B_z) = K \cap [\{ x \in R^n : (x, z) > 0\} \cup \{0\}] = (K \sim F) \cup \{0\}$.

Now consider the case when F is a nonexposed face of K. Construct a chain of faces F_i of K such that $F = F_0 \trianglelefteq F_1 \trianglelefteq F_2 \trianglelefteq \cdots \trianglelefteq F_k = K$, $k \geqslant 2$, such that each F_i is an exposed face of F_{i+1} (this is the case if F_i is a maximal face of F_{i+1}). From the first part of our proof, for each i, there exists a compact convex set C_i in $\operatorname{span} F_{i+1}$ containing 0 such that $\operatorname{cone}(0, C_i) = (F_{i+1} \sim F_i) \cup \{0\}$. Let $C = \operatorname{conv}(\bigcup_{i=0}^{k-1} C_i)$. Then C is a compact convex set containing 0 with the required property, because

$$\begin{aligned}
\operatorname{cone}(0, C) &= \operatorname{cone}\left(0, \operatorname{conv}\left(\bigcup_{i=1}^{k-1} C_i\right)\right) \\
&= \operatorname{cone}(0, C_0) + \operatorname{cone}(0, C_1) + \cdots + \operatorname{cone}(0, C_{k-1}) \\
&= [(F_1 \sim F) \cup \{0\}] + [(F_2 \sim F_1) \cup \{0\}] + \cdots \\
&\quad + [(K \sim F_{k-1}) \cup \{0\}] \\
&= (K \sim F) \cup \{0\}.
\end{aligned}$$

■

6. OPEN PROBLEMS

Problem 1. Is it true that for any cone K in R^n, there exists a compact convex set C in R^n, which contains the zero vector, such that $\operatorname{cone}(0, C) = K$?

We still seem to be remote from the solution of this question. For the simplest types of cones, using the results of Section 5, we can always find the required convex sets. The following example may be noteworthy.

EXAMPLE 6.1. Let K_1 be the ice-cream cone in R^3, i.e. the cone $\{(\xi_1, \xi_2, \xi_3) \in R^3 : (\xi_1^2 + \xi_2^2)^{1/2} \leqslant \xi_3\}$, and let K be the cone in R^3 given by $K = K_1 \sim \{\alpha(\cos 2\pi\xi, \sin 2\pi\xi, 1) : \alpha > 0 \text{ and } \xi \text{ is an irrational number in } [0,1)\}$. We are required to find a compact convex set C such that $0 \in C$ and $\operatorname{cone}(0, C) = K$. Note that $\operatorname{int} K = \operatorname{int} K_1$, and that $K = \operatorname{int} K_1 \cup \bigcup K_\xi$, where K_ξ is the 1-dimensional boundary ray $\{\alpha(\cos 2\pi\xi, \sin 2\pi\xi, 1) : \alpha \geqslant 0\}$ of K_1 and the second union on the right side is taken over all possible rationals ξ in $[0,1)$. Eumerate the rationals in $[0,1)$ in some fashion: $\xi_1, \xi_2, \ldots$. By Proposition 5.3, there exists a compact convex set C_0 in R^3 such that $\operatorname{cone}(0, C_0) = \operatorname{int} K_1 \cup \{0\}$. For each n, let $C_n = \{\alpha(\cos 2\pi\xi_n, \sin 2\pi\xi_n, 1) : 0 \leqslant \alpha \leqslant 1/n\}$. Note that $\operatorname{cone}(0, C_n) = K_{\xi_n}$. It is not difficult to show that the set $\bigcup_{n=0}^{\infty} C_n$ is closed and bounded. Now let $C = \operatorname{conv}(\bigcup_{n=0}^{\infty} C_n)$. Then C is a compact convex set, and by the first part of Remark 5.2, C has the required property.

However, if K_2 is the cone $K_1 \sim \{\alpha(\cos 2\pi\xi, \sin 2\pi\xi, 1) : \alpha > 0 \text{ and } \xi \text{ is a rational number in } [0,1)\}$, then we have not been able to decide whether there is a compact convex set whose cone at the origin is equal to K_2.

Given a property (P) about a cone, it is natural to ask the following questions:

(i) If K possesses (P), does it follow that F also possesses (P) for each face F of K?

(ii) If K possesses (P), does it follow that $\operatorname{cone}(y, K)$ also possesses (P) for each vector $y \in K$?

The results of Section 4 show that the answer to the second question is in the affirmative, if (P) is one of the following properties: the duality operator being injective, the duality operator being surjective, or the duality operator d_K satisfying the following property: for any $F, G \trianglelefteq K$, $d_K(F \wedge G) = d_K(F) \vee d_K(G)$. If (P) is the property that the duality operator is injective, then it is not difficult to show that the answer to question (i) is also in the affirmative. But for the other two properties, our knowledge about question (i) is incomplete.

PROBLEM 2. Let K be a proper cone in R^n. Suppose that the duality operator of K possesses the following property: for any $F, G \trianglelefteq K$, $d_K(F \wedge G) = d_K(F) \vee d_K(G)$. Does this imply that the duality operator of each face of K satisfies the corresponding property?

We do not know the answer to Problem 2 even when K is allowed to be a general cone.

PROBLEM 3. Let K be a proper cone in R^n. Does the surjectivity of d_K imply that of d_F for each face F of K?

We know the answer to Problem 3 is in the negative if K is allowed to be nonclosed. Below is an example.

EXAMPLE 6.2. Let M be the convex set given in Example 2.9, and let $C = M \sim \{(\xi_1, 0): -1 \leqslant \xi_1 \leqslant -\frac{1}{2} \text{ or } \frac{1}{2} \leqslant \xi_1 \leqslant 1\}$. Then C is also a convex set. Let K be the cone in R^3 given by $K = \{\alpha(\xi_1, \xi_2, 1): \alpha \geqslant 0 \text{ and } (\xi_1, \xi_2) \in C\}$. It is not difficult to see that d_K is bijective. Now, let $F = \{\alpha(\xi_1, 0, 1): \alpha \geqslant 0$ and $-\frac{1}{2} < \xi_1 < \frac{1}{2}\}$. Then F is a face of K. However, d_F is not surjective.

PROBLEM 4. Let K be a proper cone in R^n. Does the injectivity of d_K imply the condition "for any $F, G \trianglelefteq K$, $d_K(F \wedge G) = d_K(F) \vee d_K(G)$"?

Example 2.14 shows that the answer to Problem 4 is in the negative if K is allowed to be a general cone.

The solution of the following problem will help to solve Problems 3 or 4.

PROBLEM 5. Let K be an n-dimensional, closed, pointed cone in R^{n+1}. Can we always find a proper cone $\overline{K}$ of R^{n+1} containing K as a face such that (i) $d_{\overline{K}}$ is surjective or (ii) $d_{\overline{K}}$ is injective when d_K is injective?

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